Solving Graph Isomorphism Problems via Hyperdimensional Noise

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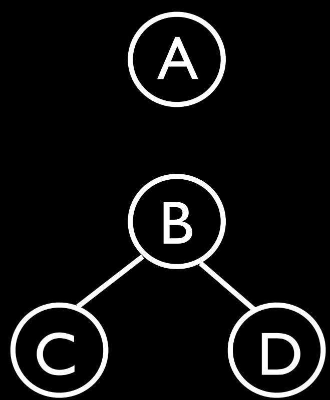
30 April 2009

Outline

1. Graphs: what & why
2. Solving graph isomorphism via Motzkin-Straus Theorem & replicator equations
3. Computing with hyperdimensional noise vectors
4. Solving graph isomorphism with hyperdimensional noise
5. Conclusions

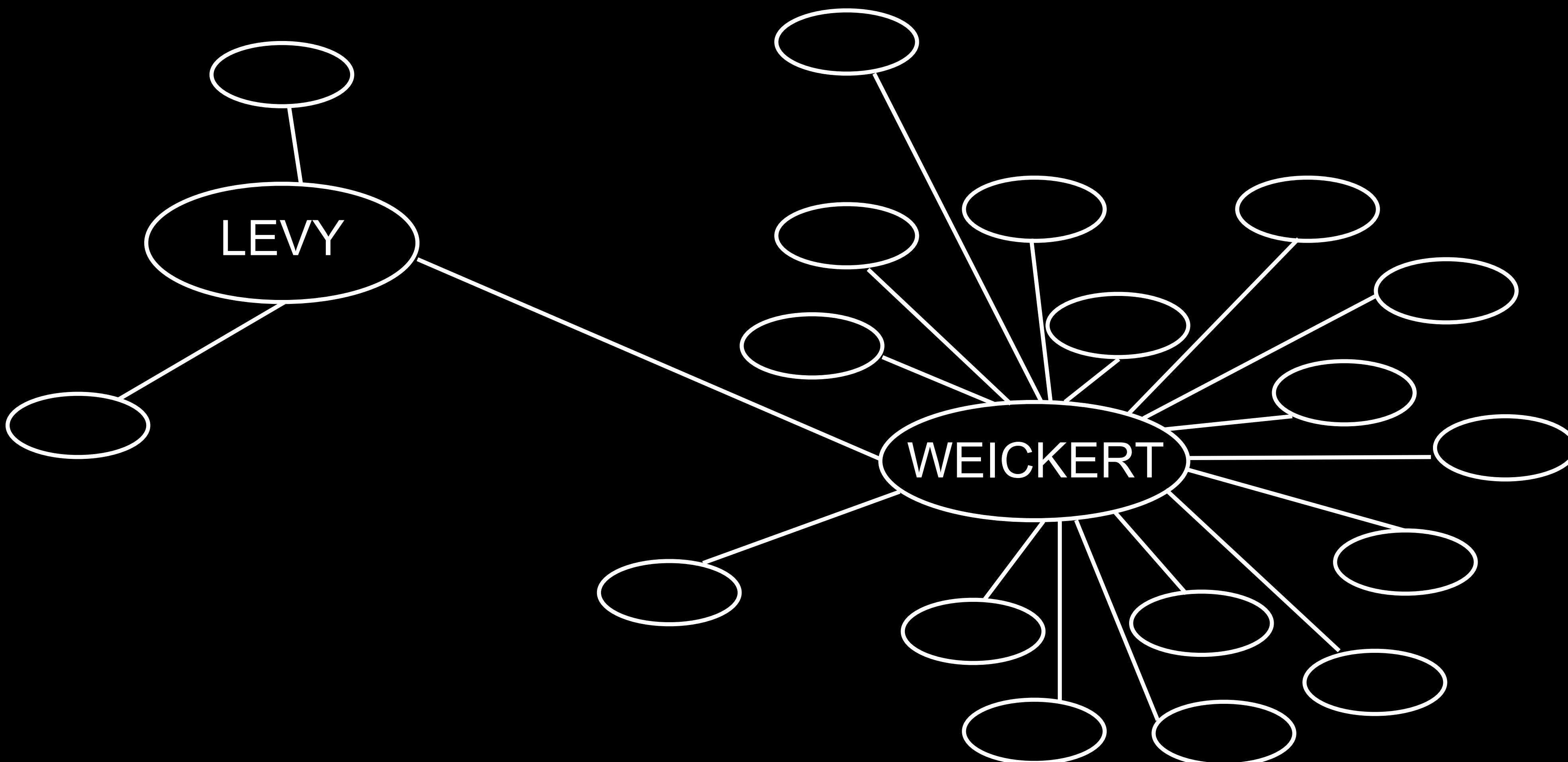
Graphs: What



- Formally, a set of vertices V and edges $E \subseteq V \times V$
- Here, V = {A, B, C, D}, E = { (B,C), (B,D) }
- I'll ignore the difference between directed & undirected graphs

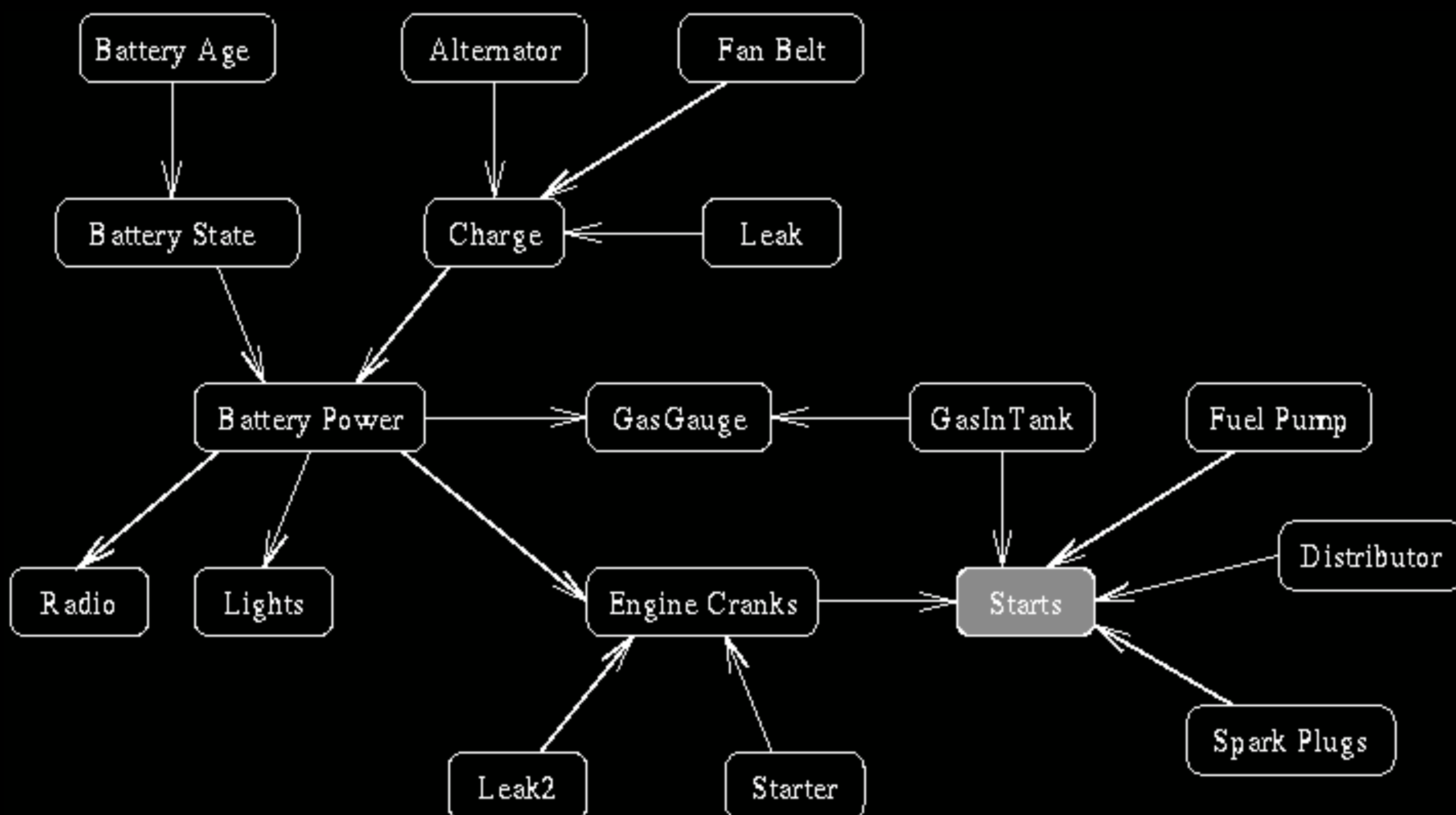
Graphs: Why

Social networks



Graphs: Why

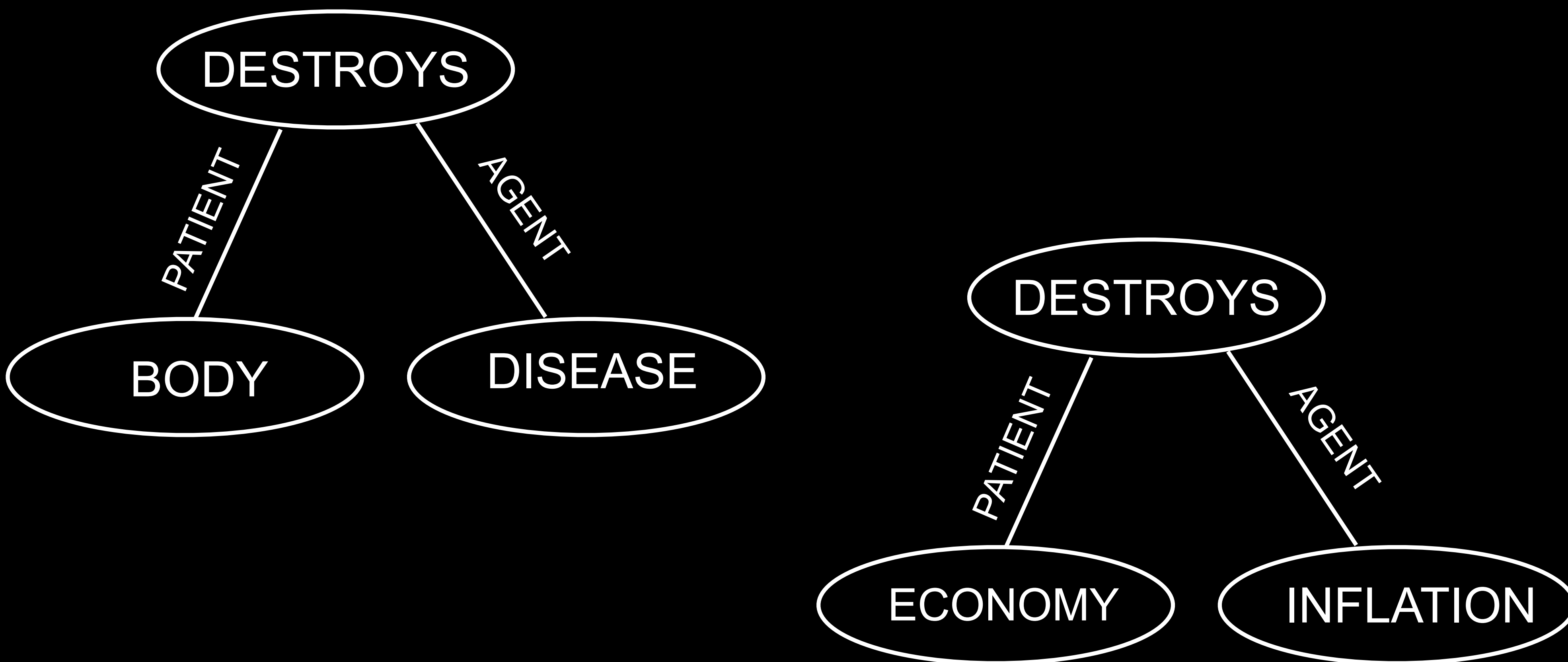
Belief Networks



http://web.engr.oregonstate.edu/~tgd/classes/539-old/programs/program4.html

Graphs: Why

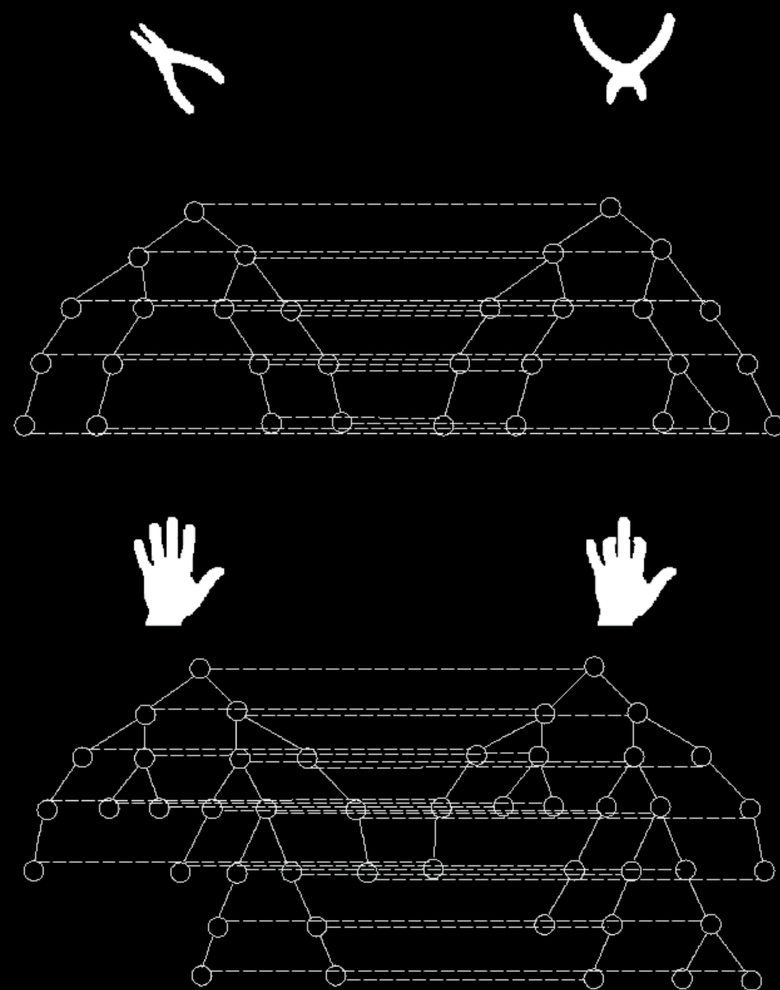
Language / Analogy



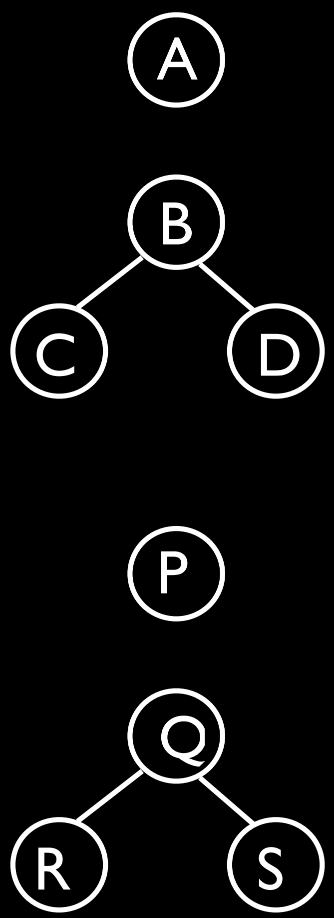
Graphs: Why

Shape Recognition

(Pelillo 1999)



The Graph Isomorphism Problem



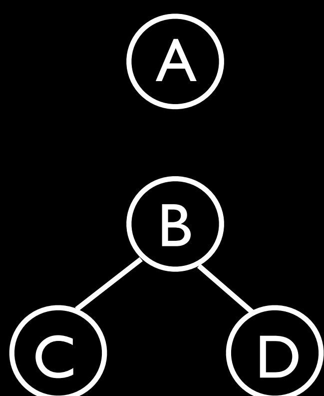
- Find good mappings to preserve edge relations:

  A=P, B=Q, C=R, D=S

  A=P, B=Q, C=S, D=R

- Believed to lie between P ("easy") and NP ("hard") problem classes

Adjacency Matrix



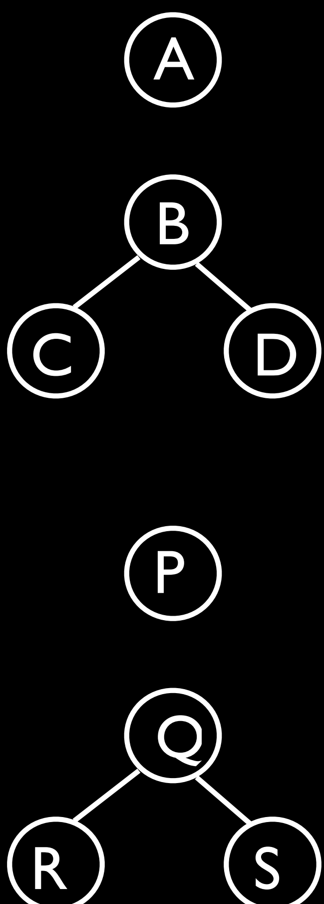
	A	B	C	D
A	0	0	0	0
B	0	0	1	1
C	0	1	0	0
D	0	1	0	0

Association Graph

Given a graph G' of size N (vertices) with an $N \times N$ adjacency matrix $A' = a''_{ij}$ and a graph G″ of size N with an $N \times N$ adjacency matrix $A'' = a''_{hk}$, their association graph G of size N^2 can be represented by an $N^2 \times N^2$ adjacency matrix $A = (a_{ih}, a_{jk})$ whose edges encode pairs of edges from G' and G″:

$$a_{ih,jk} = \begin{cases} 1 - (a'_{ij} - a''_{hk})^2 & \text{if } i \neq j \text{ and } h \neq k \\ 0 & \text{otherwise} \end{cases}$$

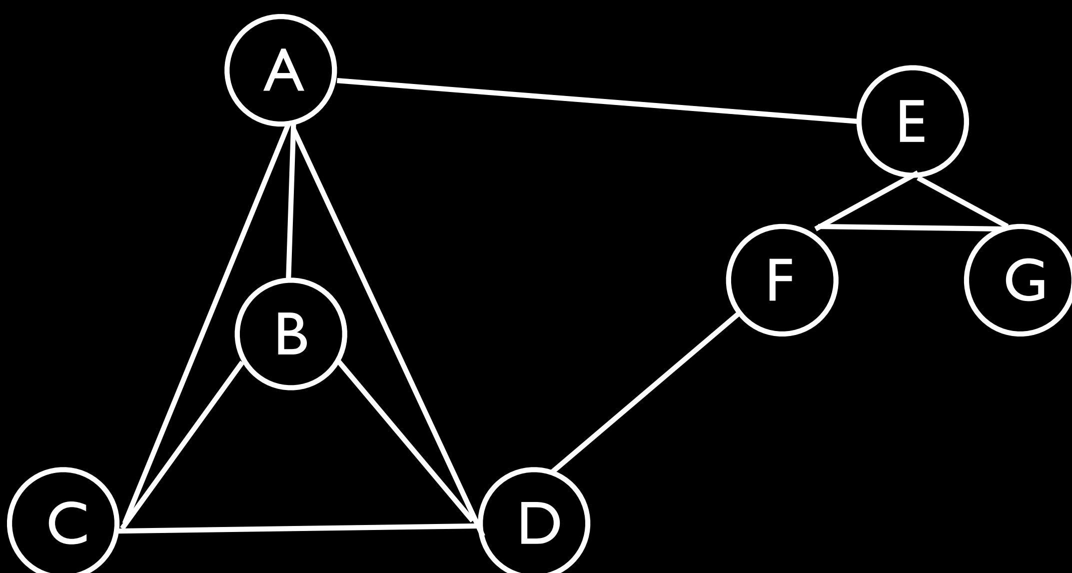
Association Graph



	AP	AQ	AR	AS	BP	BQ	BR	BS	CP	CQ	CR	CS	DP	DQ	DR	DS
AP	0	0	0	0	0	1	1	1	0	1	1	1	0	1	1	1
AQ	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
AR	0	0	0	0	1	0	0	1	1	0	0	1	1	0	0	1
AS	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0
BP	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
BQ	1	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1
BR	1	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0
BS	1	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0
CP	0	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1
CQ	1	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0
CR	1	0	0	1	0	1	0	0	0	0	0	0	1	0	0	1
CS	1	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0
DP	0	1	1	1	0	0	0	0	0	1	1	1	0	0	0	0
DQ	1	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0
DR	1	0	0	1	0	1	0	0	1	0	0	1	0	0	0	0
DS	1	0	1	0	0	1	0	0	1	0	1	0	0	0	0	0

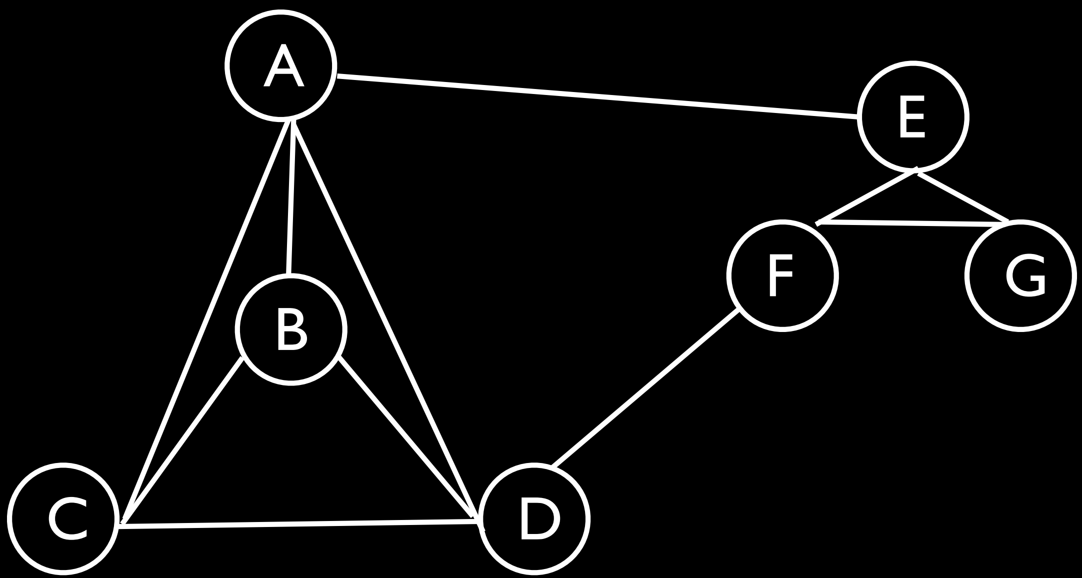
Clique

- A clique is a subset of vertices each of which is adjacent to all others.



- A,B,C,D
- E,F,G

Maximal Clique



A,B,C,D

E,F,G

Graph Isomorphism as Maximal Clique

A maximal isomorphism between two graphs *G′* and *G″* corresponds to a maximal clique in their association graph *G*.

Motzkin-Straus Theorem

Consider a graph G with adjacency matrix A, a subset C of vertices of G, and a characteristic vector x^C (indicating membership in the subset C) defined as

$$x_i^C = \begin{cases} 1/|C| & \text{if } i \in C \\ 0 & \text{otherwise} \end{cases}$$

where $|C|$ is the cardinality of C.

Motzkin-Straus Theorem

It turns out that C is a maximum clique of G iff x^C maximizes the function $f(x) = x^TAx$, where x^T is the transpose of x, $x \in \mathbf{R}^N$, $\forall i\ x_i \geq 0$, and $\Sigma_{i=1}^N x_i = 1$.

So how do we obtain x ?

Replicator Equations

Starting at some initial state (typically just $x_i = 1/N$ corresponding to all x_i being equally supported as part of the solution), x can be obtained through iterative application of the following equation:

$$x_i(t+1) = \frac{x_i(t)\pi_i(t)}{\Sigma_{j=1}^{N} x_j(t)\pi_j(t)}$$

where

$$\pi_i(t) = \Sigma_{j=1}^{N} w_{ij} x_j(t)$$

and w is a linear function of the adjacency matrix of the association graph ("evidence matrix").

Evolutionary Game Theory

$$\pi_i(t) = \Sigma_{j=1}^{N} w_{ij} x_j(t)$$

- Origins in Evolutionary Game Theory (Maynard Smith 1982)
- x_i is a strategy
- π_i is the overall payoff from that strategy
- w_{ij} is the utility of playing strategy i against strategy j

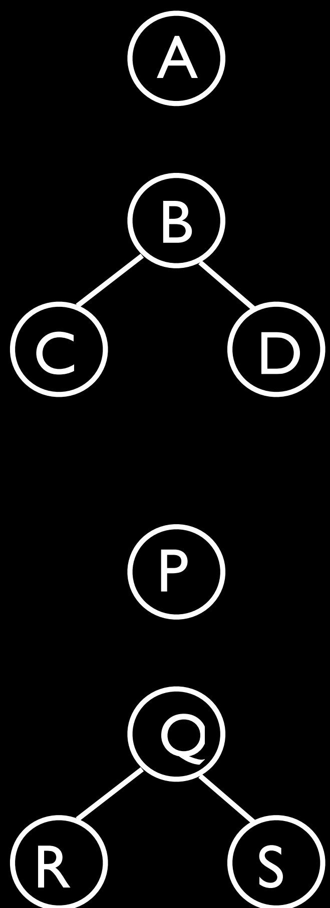
Evolutionary Game Theory

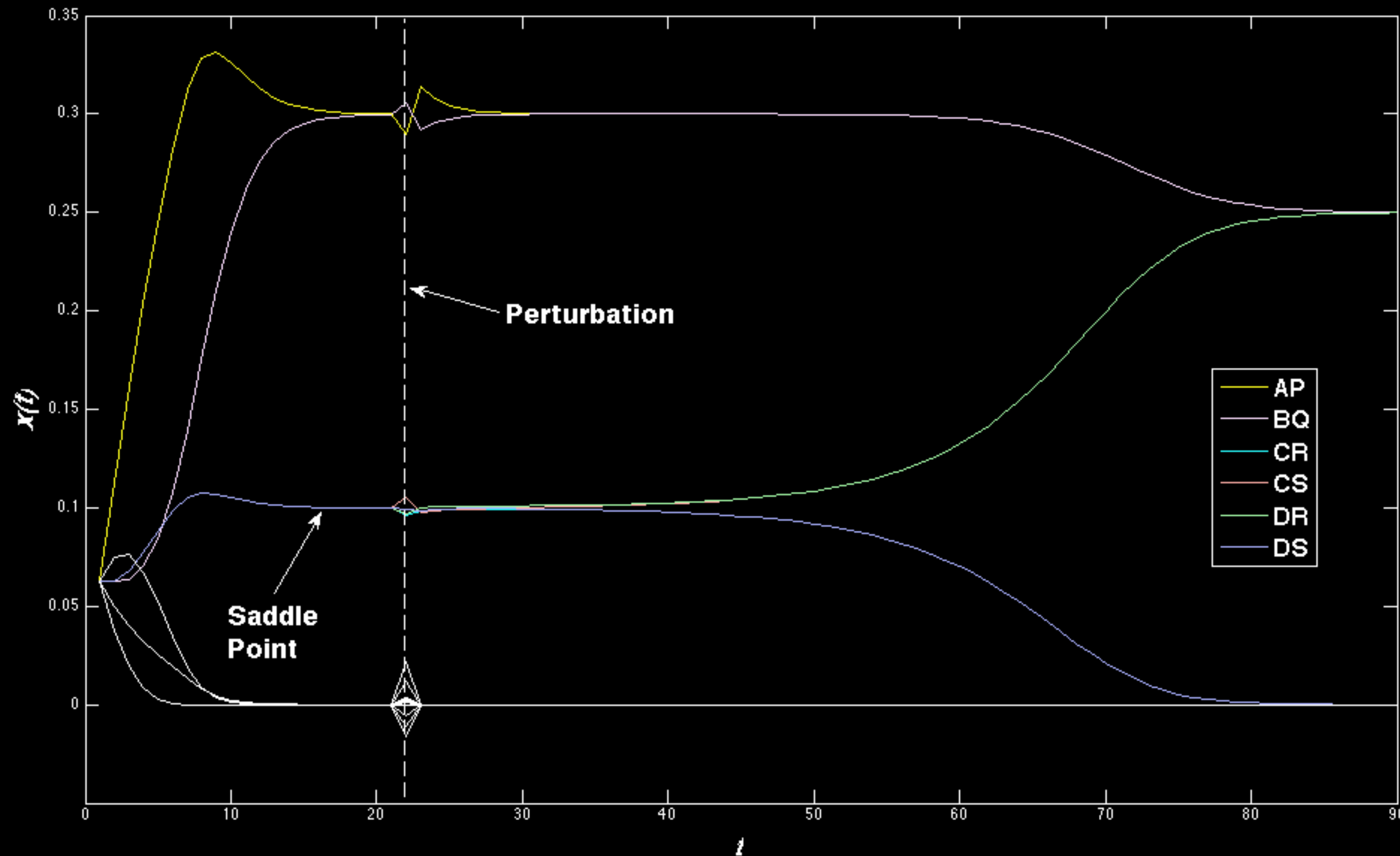
E.g. Prisoner's Dilemma

		Player B	
		Cooperate	*Defect*
Player A	*Cooperate*	A→3, B→3 Reward for mutual cooperation	A→0, B→5 Sucker's payoff and temptation to defect
	Defect	A→5, B→0 Temptation to defect and sucker's payoff	A→1, B→1 Punishment for mutual defection

http://www.jyi.org/volumes/volume11/issue3/images/hourigan_1.gif

Sample Run


A
B
C
D
P
Q
R
S


Perturbation
Saddle Point
x(t)
t
AP
BQ
CR
CS
DR
DS
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
10
20
30
40
50
60
70
80
90

Part II

Hyperdimensional Noise Vectors

- Replicator equations are biologically motivated (cf. Hull 1989 on General Selection)

- But high-precision, low-dimensional coding (*a là* Motzkin-Straus) is biologically implausible

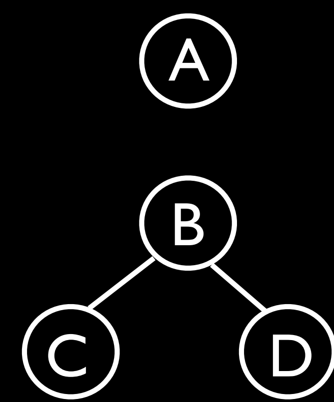
Hyperdimensional Noise Vectors

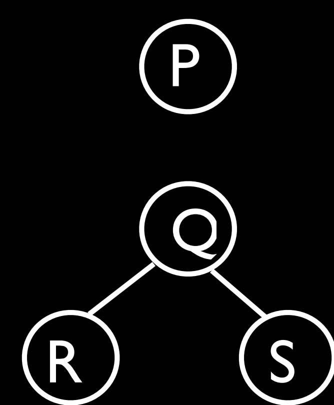
- Consider high-dimensional vectors (10,000 or more elements) randomly chosen from {+1,–1}.
- Nice properties
  - Lots of mutually orthogonal vectors, astronomical number of nearly-orthogonal
  - Highly robust to degradation
  - Each vector is its own multiplicative inverse (elementwise)

Hyperdimensional Noise Vectors

- We represent each vertex as one such vector
- Edge between A and B: A*B
- Set of edges: B*C + B*D
- Set of mappings:

  A*P + B*Q + C*R + D*S
- A “distributed data structure”





Hyperdimensional Noise Vectors

- Set of mappings:

  A*P + B*Q + C*R + D*S

- Who is associated with A?

  A*(A*P + B*Q + C*R + D*S)

  = A*A*P + A*B*Q + A*C*R +A*D*S

  = P + “junk” (via distance metric)

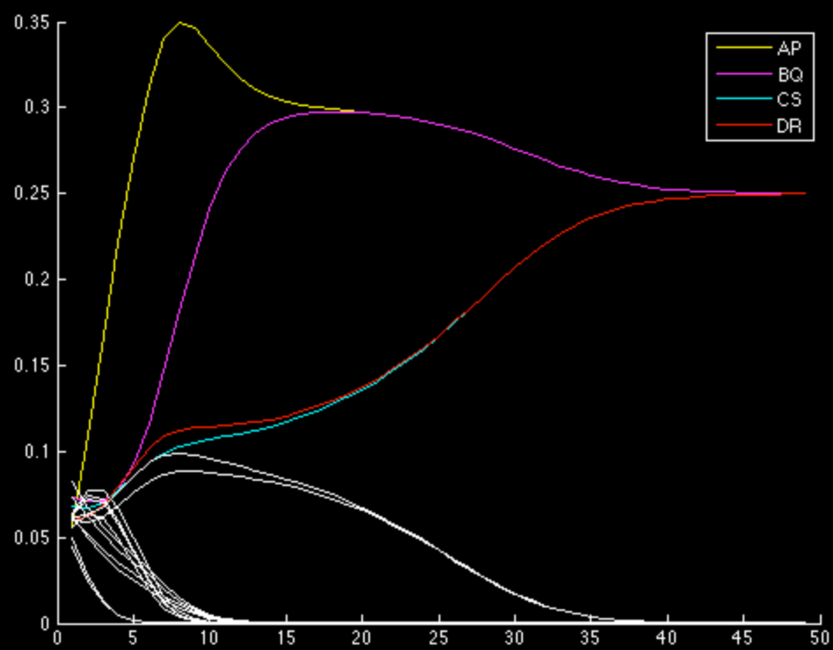
Replicator Equations with Hyperdimensional Vectors

- Possibilities *x*: A*P + A*Q + A*R + A*S + ... + D*S
- Evidence *w*: A*B*P*Q + A*B*P*R +...+ B*C*Q*R + .. + C*D*R*S

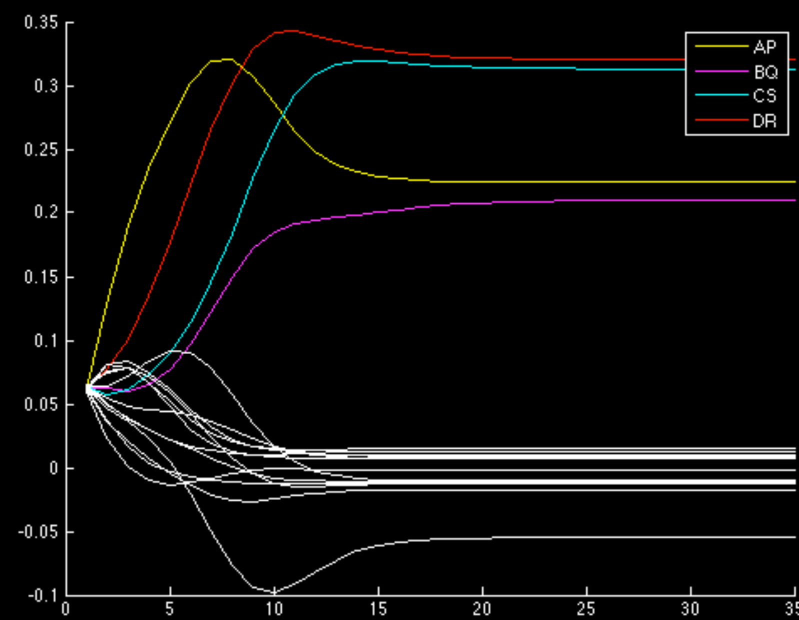
  *x***w* = A*Q + B*R + ... + A*P + ... + D*S

- NOTE: *x***w* is a single "holistic" vector operation!
- Glossed over: *w**π should reinforce, not self-cancel

Results



Pelillo (1999)



Levy & Gayler (submitted)

Conclusions

- Motivating question (P. Kanerva):

  [W]hat kinds of things suggested by the architecture of the brain, if we modeled them mathematically, could give some properties that we associate with mind?

- Difficult, highly interdisciplinary work, best done at a place like W&L

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